# Three different views on barrier functions in conic optimization


**Roland Hildebrand**

Moscow Institute of Physics and Technology
Institutskiy per. 9, 141701 Dolgoprudny, Russian Federation
Skolkovo Institute of Science and Technology
Bol'shoy blvd 30, bldg 1, 121205 Moscow, Russian Federation
khildebrand.r@mipt.ru,R.Hildebrand@skoltech.ru


INTERNATIONAL CONFERENCE ON COMPUTATIONAL OPTIMIZATION

Skolkovo Institute of Science and Technology


Self-concordant barriers studied in conic optimization correspond to objects in other branches of mathematics: centro-affine hypersurface immersions in affine differential geometry and Lagrangian submanifolds in para-Kähler geometry.


## 1. Conic programs, barriers, interior-point methods

In conic optimization problems of the form

$$\min_{x\in K}\langle c,x\rangle: \quad Ax=b \tag{1}$$

are considered, where $K\subset\mathbb{R}^n$ is a regular convex cone. The dual program is of the form

$$\max_{s\in K^*,y}\langle b,y\rangle: \ s+A^Ty=c, \tag{2}$$

defined over the dual cone $K^*=\{s\in\mathbb{R}_n \mid \langle x,s\rangle\geq 0 \quad \forall\, x\in K\}$. Here $\mathbb{R}_n=(\mathbb{R}^n)^*$.
Most conic programs solved in practice are defined over symmetric cones.

**Definition 1** *A regular convex cone is called symmetric if it is homogeneous and self-dual.*

Conic programs are solved by interior-point methods, which need a computable self-concordant barrier for running [1].

**Definition 2 (Nesterov, Nemirovski 1994)** *Let $K\subset\mathbb{R}^n$ be a regular convex cone. A (self-concordant logarithmically homogeneous) barrier on $K$ is a smooth function $F: K^o\to\mathbb{R}$ on the interior of $K$ such that*

- $F(\alpha x)=-\nu\log\alpha+F(x)$ *(logarithmic homogeneity)*
- $F''(x)\succ 0$ *(convexity)*
- $\lim_{x\to\partial K}F(x)=+\infty$ *(boundary behaviour)*
- $|F'''(x)[h,h,h]|\leq 2(F''(x)[h,h])^{3/2}$ *(self-concordance)*

*for all $\alpha>0$, $x\in K^o$, and tangent vectors $h$ at $x$. $\nu$ is called the barrier parameter.*

**Theorem 1 (Nesterov, Nemirovski 1994)** *Let $K\subset\mathbb{R}^n$ be a regular convex cone and $F: K^o\to\mathbb{R}$ a barrier on $K$ with parameter $\nu$. Then the Legendre transform*

$$F_*(s)=\sup_{x\in K}(\langle x,-s\rangle-F(x))$$

*is a self-concordant barrier on $K^*$ with parameter $\nu$.*

On symmetric cones self-scaled barriers exist which accelerate interior-point methods.
The idea of the method consists in approximately solving auxiliary problems of the form

$$\min_x(\tau\langle c,x\rangle+F(x)): \quad Ax=b; \qquad \max_{s,y}(\tau\langle b,y\rangle-F_*(s)): \quad s+A^Ty=c.$$

Here $\tau>0$ is a parameter. The solutions $(x^*(\tau),s^*(\tau),y^*(\tau))$ satisfy the relations

$$Ax^*=b, \quad s^*+A^Ty^*=c, \quad s^*=-\tau^{-1}F'(x^*), \quad x^*=-\tau^{-1}F_*'(s^*) \tag{3}$$

and form the primal-dual central path of the problem. The auxiliary problems are solved by a Newton-like method, where the nonlinear objectives are linearized at a scaling point $w$ satisfying the condition

$$F''(x)w=s.$$

Here $x,s$ is the current iterate of the method.

## 2. Affine differential geometry

Affine invariants of hypersurfaces $M\subset\mathbb{R}^n$ are studied by affine differential geometry. Let a transversal vector field $\xi$ be defined on $M$. The acceleration of curves defines a quadratic form $h$ and a cubic form $C$ on $M$. If the surface is locally strongly convex, $h$ serves as a metric (see Fig. 1).

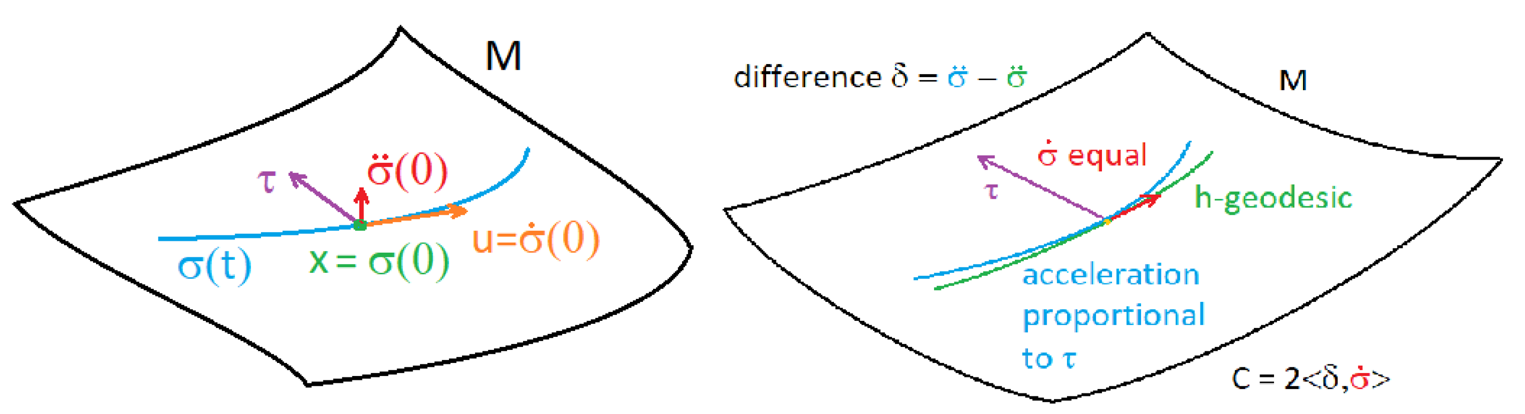


**Figure 1:** *Left: The acceleration $\ddot\sigma$ is decomposed into a tangential part and a part parallel to $\xi$. The latter is quadratic in the tangent vector $\dot\sigma$ and defines a quadratic form $h$ which serves as metric. Right: The difference in acceleration between a metric geodesic and a curve which accelerates always in the direction of $\xi$ defines a cubic form $C$.*

**Theorem 2 (Fundamental theorem of affine differential geometry)** *If the metric $h$ and the cubic form $C$ are known on $M$, then the immersion of $M$ into $\mathbb{R}^n$ can be recovered up to an affine transformation of $\mathbb{R}^n$.*

Several choices of the transversal field $\xi$ are commonly studied [2]:

- $\xi(x)=const$: graph immersions
- $\xi(x)=x$: centro-affine immersions
- $\xi$ is affine normal: Blaschke immersions

The first two are artificial choices, the third uses the structure of the surface itself to define $\xi$.
Objects studied in affine differential geometry:

- improper affine spheres: affine normal is constant
- proper affine spheres: affine normal is centro-affine

**Theorem 3 (Calabi theorem, 1976–1992)** *Let $K\subset R^n$ be a regular convex cone. Then there exists a unique complete proper affine sphere with centre at the origin which is asymptotic to the boundary of $K$.*

## 3. Equivalences barriers — affine differential geometry

Log-homogeneous barriers correspond to centro-affine geometry:

- centro-affine metric on level surfaces $M=\{x\mid F(x)=const\}$ coincides with $\nu^{-1}F''|_M$
- centro-affine cubic form on $M$ coincides with restriction $\nu^{-1}F'''|_M$
- self-concordance means bounded-ness of cubic form measured in the metric

The affine spheres postulated by the Calabi theorem lead to an universal construction of self-concordant barriers on arbitrary cones [3].

**Theorem 4 (H. 2014; Fox 2015)** *Let $K\subset R^n$ be a regular convex cone, set $\nu=n$. Let $M\subset K^o$ be the proper affine sphere which is asymptotic to $\partial K$. Then the log-homogeneous function $F$ defined from $M$ as level surface $M=\{x\mid F(x)=0\}$ is a self-concordant barrier, the canonical barrier.*

The self-scaled barriers can also be described in geometric terms [4].

**Theorem 5 (H. 2014)** *Let $K$ be a regular convex cone and $F$ a log-homogeneous barrier on $K$. Then the following are equivalent:*

- *$K$ is symmetric and $F$ is self-scaled*
- *the centro-affine cubic form $C$ on the level surface $M=\{x\mid F(x)=const\}$ is parallel (has a vanishing covariant derivative) with respect to the centro-affine metric $h$*

The parallelism condition can be expressed as a PDE:

$$F_{,\alpha\beta\gamma\delta}=\frac{1}{2}\sum_{\rho\sigma}F^{,\rho\sigma}\left(F_{,\alpha\beta\rho}F_{,\gamma\delta\sigma}+F_{,\alpha\gamma\rho}F_{,\beta\delta\sigma}+F_{,\alpha\delta\rho}F_{,\beta\gamma\sigma}\right)$$

This yields a local characterization of self-scaledness.

## 4. Equivalences barriers — Lagrangian submanifolds

On $\mathbb{R}^n$ there does not exist a natural metric. However, the para-Kähler space $P=\mathbb{R}^n\times\mathbb{R}_n$ is equipped with both a pseudo-Riemannian metric $g$ and a symplectic form $\omega$,

$$g((u_P,u_D),(v_P,v_D)):=\frac{1}{2}(\langle u_P,v_D\rangle+\langle v_P,u_D\rangle),\ \omega((u_P,u_D),(v_P,v_D)):=\frac{1}{2}(\langle u_P,v_D\rangle-\langle v_P,u_D\rangle).$$

Gradient graphs $\Gamma=\{(x,F'(x))\mid x\in D\}$ of functions $F:\mathbb{R}^n\supset D\to\mathbb{R}$ are then Lagrangian submanifolds of $P$, i.e., $\omega|_M=0$. They can also be considered as gradient graphs of the Legendre dual $F_*$, with the role of the primal and dual factor reversed. Moreover, we have the following result.

**Theorem 6** *Let $D\subset\mathbb{R}^n$ be a domain, $F:D\to\mathbb{R}$ a $C^3$ function. Let $\Gamma\subset P$ be the gradient graph of $F$. Then*

- *the manifold $\Gamma$ equipped with the submanifold metric from $P$ is isometric to the domain $D$ equipped with the Hessian metric $F''$*
- *the extrinsic curvature (deviation from a geodesic submanifold) of $\Gamma$ as an embedding into $P$ is proportional to the derivative $F'''$*

Consequently, the self-concordant functions are characterized by a bounded deviation of $\Gamma$ from a geodesic submanifold.
The linear equality conditions of problems (1),(2) define an $n$-dimensional affine subspace $\mathcal{A}$ in the $2n$-dimensional product space $P$. Then the central path defined by (3) is given by the intersection

$$\mathcal{A}\cap(\mathbb{R}_{++}\cdot\Gamma),$$

and the scaling point is characterized by the following condition.

**Theorem 7** *Let $x,s$ be the current iterate. Then the point $(w,-F'(w))$ on the gradient graph corresponding to the scaling point $w$ is a critical point of the distance function from the pair $(x,s)\in P$ on $\Gamma$.*

The gradient graph of the canonical barrier also has a geometric characterization.

**Theorem 8** *Let $K\subset\mathbb{R}^n$ be a regular convex cone and $F$ a barrier on $K$. Then the following are equivalent:*

- *$F$ is the canonical barrier, up to an affine scaling*
- *the gradient graph $\Gamma$ of $F$ is a minimal surface in $P$*

A more developed theory exists for 3D cones [5]. For several families of 3D non-symmetric cones the canonical barrier has been computed [6].

## References


[1] Y. Nesterov and A. Nemirovski. *Interior-Point Polynomial Algorithms in Convex Programming.* SIAM, 1994.

[2] K. Nomizu and T. Sasaki. *Affine differential geometry.* Cambridge Univ. Press, 1994.

[3] R. Hildebrand. Canonical barriers on convex cones. *Math. Oper. Res.* 39(3):841–850, 2014.

[4] R. Hildebrand. Hessian Potentials with Parallel Derivatives. *Res. Math.* 65:399–413, 2014.

[5] J. Dorfmeister, R. Hildebrand and S. Kobayashi. Half-Dimensional Immersions into the Para-Complex Projective Space and Ruh–Vilms Type Theorems. *Res. Math.* 79:245, 2024.

[6] R. Hildebrand. Analytic formulas for complete hyperbolic affine spheres. *Contrib. Algebra Geom.* 55:497–520, 2014.