Primal-Dual Gradient Methods for Searching Network Equilibria in Combined Models with Nested Choice Structure and Capacity Constraints

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Overview

We consider the problem of forecasting travel demand in a transportation system. This problem arises in

- the management of infrastructure development
- the land use planning
- the policymaking for maintaining sustainable transportation systems.

1. Traditional Four-Step Procedure

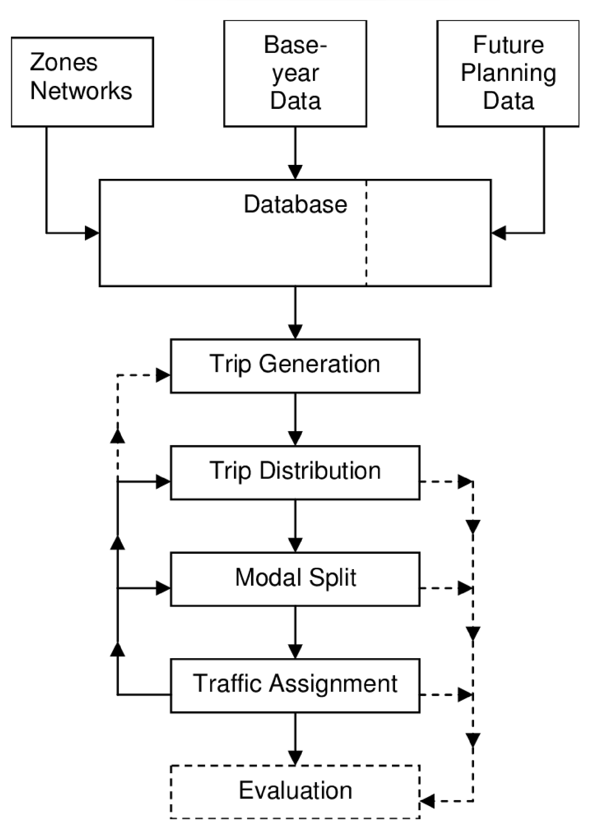


Figure: The classic four-stage transport model

Trip Distribution

- L_i number of departures from zone $i \in O$
- W_j number of arrivals to zone $j \in D$
- T_{ij} generalized cost for travelling from zone i to j
- d_{ij} number of trips from zone i to j
- γ calibration parameter

Entropy Maximizing Model

$$-\sum_{i,j} d_{ij}T_{ij} - \underbrace{\frac{1}{\gamma}\sum_{i,j} d_{ij}\ln d_{ij}}_{H(d)} \longrightarrow \max_{\substack{d_{ij}:\, d_{ij}\geq 0,\\ \sum_j d_{ij}=L_i,\\ \sum_i d_{ij}=W_j}}$$

Modal Split

- d_{ij}^{am} number of trips from zone i to j by mode m and agent a
- T_{ij}^{m} generalized cost of travelling from i to j by mode m
- $M(a)$ set of modes available to type a agents
- α_{am}, β_{am} calibration parameters

The Multinomial Logit Model (MNL)

$$\frac{d_{ij}^{am}}{d_{ij}^{a}} = \frac{\exp(-\alpha_{am}T_{ij}^{m}+\beta_{am})}{\sum_{m\in M(a)}\exp(-\alpha_{am}T_{ij}^{m}+\beta_{am})}$$

Traffic assignment

Beckmann model

x^* is an equilibrium state if and only if x^* is a minimum of the potential function:

$$\Psi(f(x)) = \sum_{e\in E}\int_0^{f_e}\tau_e(z)dz \to \min_{f=\Theta x:\, x\in X},$$

BPR-function:

$$\tau_e(f_e) = \bar{t}_e\left(1+\rho\left(\frac{f_e}{\bar{f}_e}\right)^{\frac{1}{\mu}}\right),\ \rho=0.15,\ \mu=0.25,$$

where $\bar{t}_e$ – free flow time, $\bar{f}_e$[veh/hour] – link capacity. Dual problem:

$$Q(t) = \sum_{ij\in OD} d_{ij}T_{ij}(t) - \underbrace{\sum_{e\in\mathcal{E}}\sigma_e^*(t_e)}_{h(t)} \longrightarrow \max_{t\geq\bar{t}} \qquad (1)$$

where $\sigma_e^*(t_e)$ is the Fenchel conjugate function of $\sigma_e(f_e)$, $e\in E$.

Stable dynamics model [3]

$$\tau_e(f_e) = \begin{cases}\bar{t}_e, & 0\leq f_e<\bar{f}_e,\\ [\bar{t}_e,\infty], & f_e=\bar{f}_e,\\ +\infty, & f_e>\bar{f}_e.\end{cases}$$

The pair (f^*,t^*) is an equilibrium if and only if it is a solution of the saddle-point problem

$$S(f(x),t) = \langle t,f\rangle - \underbrace{\langle t-\bar{t},\bar{f}\rangle}_{h(t)} \longrightarrow \min_{\substack{f=\Theta x:\\ x\in X}}\max_{t\geq\bar{t}},$$

2. Refined: Network equilibrium model (NE)

According to [1], the combined distributionmodal splitassignment problem can be formulated as follows:

$$P_3(f,d) = \Psi(f)+H(d) \to \min_{\substack{f=\Theta x,\, x\in X(d)\\ d\in\Pi'(l,w)}}, \qquad (P3)$$

where

$$H(d) = \sum_{i,j,r,a}\frac{1}{\gamma_r}d_{ij}^{ra}\ln d_{ij}^{ra} + \sum_{i,j,r,a,m}\frac{1}{\alpha_a}d_{ij}^{ram}\left(\ln\left(\frac{d_{ij}^{ram}}{d_{ij}^{ra}}\right)+\beta_{am}\right).$$

The saddle-point problem:

$$S_3(d,t) = \underbrace{\sum_{i,j,r,a} d_{ij}^{ra}T_{ij}^{a}(t) + \sum_{i,j,r,a}\frac{1}{\gamma_r}d_{ij}^{ra}\ln d_{ij}^{ra}}_{E(d,T(t))} - h(t) \to \min_{d\in\Pi(l,w)}\max_{t\geq\bar{t}}, \qquad (S3)$$

where $T_{ij}^{a}(t) = -\frac{1}{\alpha_a}\ln\left(\sum_m\exp\left(-\alpha_a T_{ij}^{m}(t)-\beta_{am}\right)\right)$, $T_{ij}^{m}(t)$ is the minimal cost of the path from $i\in O$ to $j\in D$ with the links cost $t_e+c_e^m$.

The dual problem is

$$D_3(t) = \underbrace{\min_{d\in\Pi(l,w)} E(d,T(t)) - h(t)}_{-\Phi(t)} \longrightarrow \max_{t\geq\bar{t}}, \qquad (D3)$$

3. Evans Algorithm

1. For each edge $e\in E$ calculate the costs τ_e that correspond to the flows f_e^k.
2. For each origin vertex $i\in O$ find the minimum T_{ij} of travelling to each destination $j\in D$ and choose a minimum cost route from i to j.
3. Find a new set of trip distributions q_{ij}, given the new T_{ij} costs.
4. Assign the new demands q_{ij} to the minimum cost routes chosen in Step 2 to obtain a new flow vector $y\in\mathbb{R}^{|E|}$
5. Find the linear combination $(1-\lambda)(f^k,d^k)+\lambda(y,q)$, $0\leq\lambda\leq 1$, of (f^k,d^k) and (y,q) that minimizes the objective function P_3 and

4. Dual method for NE problem

Here we used the following notations:

$$\phi_0(t) = \frac{1}{2}\left\|t-t^0\right\|_2^2,$$

$$\phi_{k+1}(t) = \phi_k(t) + \alpha_{k+1}\left[\tilde{\Phi}(y^{k+1}) + \left\langle\tilde{\nabla}\Phi(y^{k+1}), t-y^{k+1}\right\rangle + h(t)\right].$$

Note that we did not specify the stopping criterion as it can be different for different models

Algorithm Universal Method of Similar Triangles

Require: $L_0>0$, starting point t^0, accuracy $\varepsilon>0$

$u^0 := t^0$, $A_0 := 0$, $k := 0$
repeat
  $L_{k+1} := L_k/2$
  while true do
    $\alpha_{k+1} := \frac{1}{2L_{k+1}} + \sqrt{\frac{1}{4L_{k+1}^2}+\frac{A_k}{L_{k+1}}}$, $A_{k+1} := A_k+\alpha_{k+1}$
    $y^{k+1} := \frac{\alpha_{k+1}u^k+A_kt^k}{A_{k+1}}$
    $u^{k+1} := \arg\min_{t\in\mathrm{dom}\,h}\phi_{k+1}(t)$
    $t^{k+1} := \frac{\alpha_{k+1}u^{k+1}+A_kt^k}{A_{k+1}}$
    if $\tilde{\Phi}(t^{k+1}) \leq \tilde{\Phi}(y^{k+1}) + \left\langle\tilde{\nabla}\Phi(y^{k+1}), t^{k+1}-y^{k+1}\right\rangle + \frac{L_{k+1}}{2}\left\|t^{k+1}-y^{k+1}\right\|_2^2 + \frac{\alpha_{k+1}}{2A_{k+1}}\varepsilon$ **then**

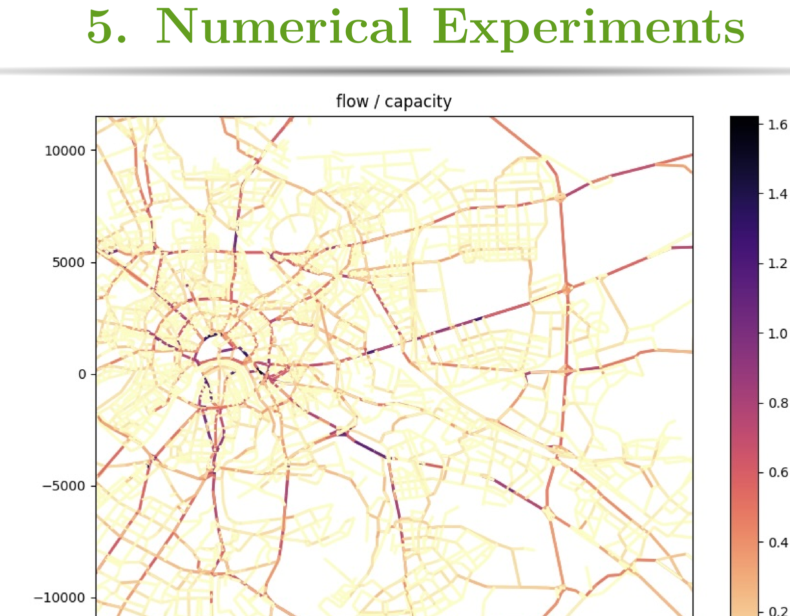
    else
      $L_{k+1} := 2L_{k+1}$
    end if
  end while
  $k := k+1$
until Stopping criterion is fulfilled

Main Contributions

- ◇ We propose a way to **solve** the dual problem of **the nested combined model** of [1] with a universal accelerated gradient method USTM [2];
- ◇ We **extend the nested combined model** to the case of capacitated networks: namely, we propose NE with the stable dynamics [3] traffic assignment model;
- ◇ We provide **theoretical upper bounds** on the complexity of searching network equilibrium by the USTM algorithm.

publication ref. :

5. Numerical Experiments



- road network: 63073 nodes 94546 arcs (roads, permitted turns at the intersections)
- 1420 transportation zones
- trip purposes: home-work, home-other
- users types: car-owners, non-car-owners
- travel modes: by foot, car, and public transport

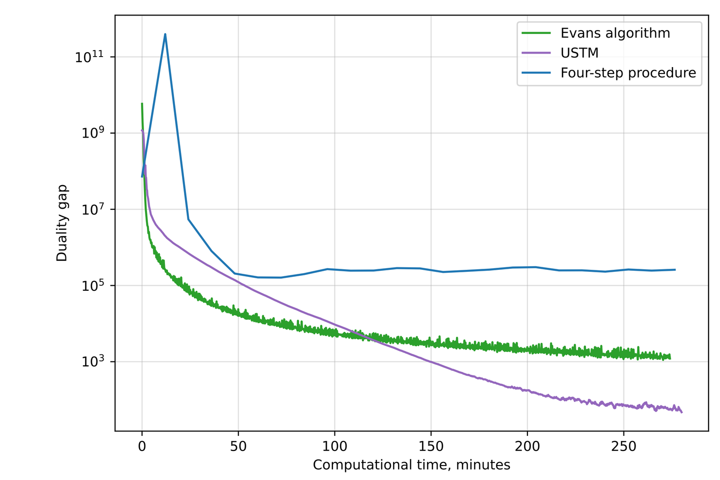


Figure: Duality gap convergence

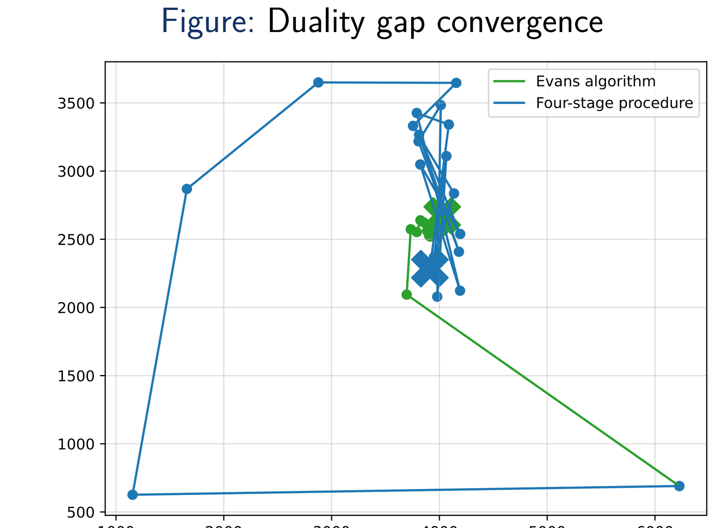


Figure: 2-Dimensional projections of d_{ij}^{m} trajectories for the Evans algorithm and the Four-stage procedure, obtained by multidimensional scaling. The trajectory of the Evans method is sparsified to 50 points. The last point is marked with a large cross

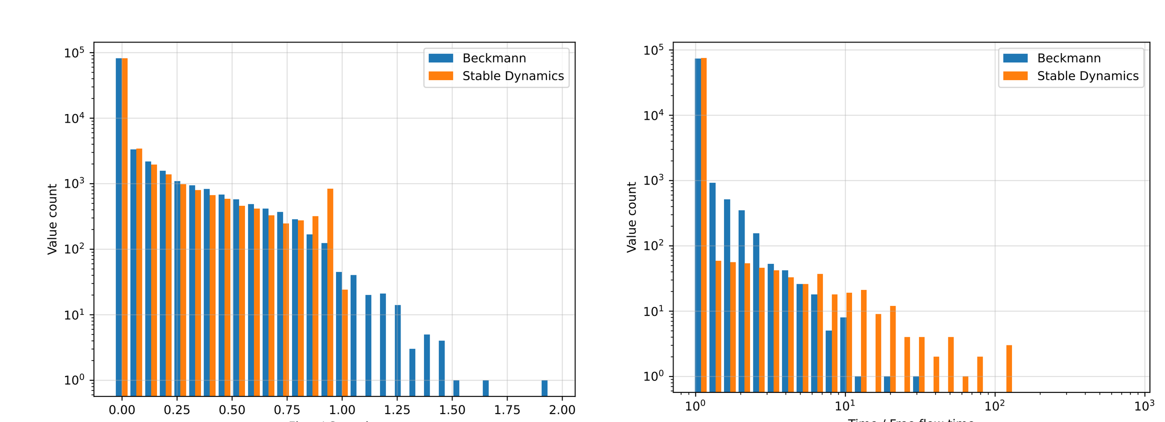


Figure: Histograms of the network load: a) histogram of the ratio of the amount of flow on the link to its capacity, b) histogram of the ratio of the travel time on the link to the travel time on the same link when it is free

8. References

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