Integration Bee, 2024 Apr. 7

Qualifying round

1) $\int \sin^2 x \; dx$ — Answer: $\frac{x}{2} - \frac{1}{4}\sin(2x) + c$

2) $\int \frac{x+2024}{x+2025} \; dx$ — Answer: $x - \ln|x+2025| + c$

3) $\int\limits_0^1 \ln\big(1+(e-1)x\big) \; dx$ — Answer: $\frac{1}{e-1}$

4) $\int \cot^2 x \; dx$ — Answer: $-x - \cot x + c$

5) $\int\limits_{-1}^{1} \sqrt{1-x^2} \; dx$ — Answer: $\frac{\pi}{2}$

6) $\int \sqrt[7]{7x+1} \; dx$ — Answer: $\frac{1}{8}(7x+1)^{8/7} + c$

7) $\int \frac{x^3}{x+1} \; dx$ — Answer: $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x - \ln|x+1| + c$

8) $\int\limits_0^{\log_5 e} 5^{x+5^x} \; dx$ — Answer: $\frac{5^e-5}{\ln^2 5}$

9) $\int \sqrt{x}\cos\sqrt{x} \; dx$ — Answer: $2(x-2)\sin\sqrt{x} + 4\sqrt{x}\cos\sqrt{x} + c$

10) $\int\limits_0^{100} [x] \; dx$, while $[x]$ is the greatest integer $\leq x$ — Answer: 4950

Playoffs

1) $\int \left(1 - \frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\ dx = \int x^{3/4}\ dx - \int x^{-5/4}\ dx = \frac{4}{7}x^{7/4} + 4x^{-1/4} + c$

2) $\int \frac{x^2\ dx}{(8x^3+27)^{2/3}} = \frac{1}{24}\int \frac{d(8x^3+27)}{(8x^3+27)^{2/3}} = \frac{1}{8}(8x^3+27)^{1/3} + c$

3) $\int \frac{dx}{x\sqrt{\ln x}} = \int \frac{d\ln x}{\sqrt{\ln x}} = 2\sqrt{\ln x} + c$

4) $\int \frac{dx}{\sin^2 x \cos^2 x} = \int \left(\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}\right)\ dx = \tan x - \cot x + c$

Final round

Task 1. Derive recurrent relation for

$$I_n = \int \tan^n x\ dx$$

Solution. $I_0 = c$, $I_1 = \int \tan x\ dx = -\ln|\cos x| + c$, for $n \geq 1$ we have

$$I_{n+1} = \int \tan^{n+1} x\ dx = \int \frac{1-\cos^2 x}{\cos^2 x}\tan^{n-1} x\ dx = \int \frac{1}{\cos^2 x}\tan^{n-1} x\ dx - \int \tan^{n-1} x\ dx =$$

$$= \int \tan^{n-1} x\ d\tan x - I_{n-1} = \frac{1}{n}\tan^n x - I_{n-1}$$

Task 2. $\int\limits_1^2 \left(\sqrt{5-x^2} + \sqrt[3]{2-x}\right)\ dx = 5\int\limits_1^2 \sqrt{1-\left(\frac{x}{\sqrt{5}}\right)^2}\ d\left(\frac{x}{\sqrt{5}}\right) - \frac{3}{4}(2-x)^{4/3}\Big|_1^2 = 5\int\limits_{\arcsin(1/\sqrt{5})}^{\arcsin(2/\sqrt{5})} \cos^2 t\ dt - (0 - \frac{3}{4}) = \frac{3}{4} + \frac{5}{2}\int\limits_{\arcsin(1/\sqrt{5})}^{\arcsin(2/\sqrt{5})} (1+\cos 2t)\ dt = \frac{3}{4} + \frac{5}{2}(\arcsin(2/\sqrt{5}) - \arcsin(1/\sqrt{5})) + \frac{5}{4}\int\limits_{\arcsin(1/\sqrt{5})}^{\arcsin(2/\sqrt{5})} \cos 2t\ d2t = \frac{3}{4} + \frac{5}{2}(\arcsin(2/\sqrt{5}) - \arcsin(1/\sqrt{5})) = \frac{3}{4} + \frac{5}{2}\arcsin\frac{3}{5}$

Task 3. $\int\limits_{-1}^{3} \big|2 - 3|x-1|\big|\ dx = \frac{20}{3}$