Integration Bee, 2023 Dec. 10th

Qualifying round

1) $\int\limits_0^1 x^\pi \ dx$ Answer: $\frac{1}{\pi+1}$

2) $\int 2^{\log_3 x} \ dx$ Answer: $\frac{x^{\log_3 6}}{\log_3 6} + c$

3) $\int\limits_{-\pi/2}^{\pi/5} \sin(10x) \ dx$ Answer: -0.2

4) $\int \frac{x}{1-x} \ dx$ Answer: $-x - \ln|1-x| + c$

5) $\int \cos^3 x \ dx$ Answer: $\sin x - \frac{1}{3}\sin^3 x + c$

6) $\int\limits_{-2}^{2} xe^{x^4} \ dx$ Answer: 0

7) $\int\limits_1^{100} \log_{10} x \ dx$ Answer: $200 - 99\log_{10} e$

8) $\int \tan^2 x \ dx$ Answer: $\tan x - x + c$

9) $\int \frac{x+3}{x^2-1} \ dx$ Answer: $2\ln|x-1| - \ln|x+1| + c$

10) $\int \frac{dx}{\sqrt{x(x+1)}}$ Answer: $2\ln|\sqrt{x} + \sqrt{x+1}| + c$

Additional qualifying round

1) $\int \frac{\cot x}{\sin x} \ dx$ Answer: $-\frac{1}{\sin x} + c$

2) $\int\limits_0^\pi \cos^4 x \ dx$ Answer: $\frac{3\pi}{8}$

3) $\int \frac{dx}{x \log_5 x}$ Answer: $\ln 5 \cdot \ln|\ln x| + c$

4) $\int x^{2023} \cdot 2024^{x^{2024}} \ dx$ Answer: $\frac{1}{\ln 2024} \cdot 2024^{x^{2024}-1} + c$

5) $\int \frac{dx}{\sqrt{9x^2 - x^4}}$ Answer: $-\frac{1}{3}\ln\left|\frac{1}{|x|} + \sqrt{\frac{1}{x^2} - \frac{1}{9}}\right| + c$

Playoffs

1) $\int\limits_0^{-3} \frac{x^3}{x+1}\ dx = \int\limits_0^{-3} \frac{x^3+1}{x+1} dx - \int\limits_0^{-3} \frac{dx}{x+1} = \int\limits_0^{-3} (x^2 - x + 1) dx - \ln|x+1|\Big|_0^{-3} = (\frac{1}{3}x^3 - \frac{1}{2}x^2 + x)\Big|_0^{-3} - (\ln 2 - \ln 1) = -9 - 4.5 - 3 - \ln 2 = -16.5 - \ln 2$

2) $\int \frac{x}{\sqrt[3]{2-7x}}\ dx = -\frac{1}{7}\int \frac{2-7x}{\sqrt[3]{2-7x}} dx + \frac{2}{7}\int \frac{dx}{\sqrt[3]{2-7x}} = \frac{1}{49}\int (2-7x)^{2/3} d(2-7x) - \frac{2}{49}\int (2-7x)^{-1/3} d(2-7x) = \frac{3}{245}(2-7x)^{5/3} - \frac{3}{49}(2-7x)^{2/3} + c$

3) $\int\limits_4^7 \frac{dx}{(1-x)(2-x)(3-x)} = \frac{1}{2}\int\limits_4^7 \frac{dx}{1-x} - \int\limits_4^7 \frac{dx}{2-x} + \frac{1}{2}\int\limits_4^7 \frac{dx}{3-x} = \frac{1}{2}\ln|1-x|\Big|_4^7 - \ln|2-x|\Big|_4^7 + \frac{1}{2}\ln|3-x|\Big|_4^7 = \frac{1}{2}\ln 2 - \ln\frac{5}{2} + \frac{1}{2}\ln 4 = \ln\left(\frac{4}{5}\sqrt{2}\right)$

4) $\int \frac{x+5}{x^2+5}\ dx = \frac{1}{2}\int \frac{2x\ dx}{x^2+5} + 5\int \frac{dx}{x^2+5} = \frac{1}{2}\ln(x^2+5) + \sqrt{5}\,\mathrm{arctg}\,\frac{x}{\sqrt{5}} + c$

5) $\int\limits_0^{16} \frac{dx}{1+\sqrt{x}} = 2\int\limits_0^{16} \frac{\sqrt{x}\ d\sqrt{x}}{1+\sqrt{x}} = 2\int\limits_0^{16} \frac{(\sqrt{x}+1)\ d\sqrt{x}}{1+\sqrt{x}} - 2\int\limits_0^{16} \frac{d\sqrt{x}}{1+\sqrt{x}} = 2\sqrt{x}\Big|_0^{16} - 2\ln(\sqrt{x}+1)\Big|_0^{16} = 8 - 2\ln 5$

6) $\int \frac{dx}{\cos(\frac{x}{3})} = 3\int \frac{\cos\frac{x}{3}\ d\frac{x}{3}}{1-\sin^2\frac{x}{3}} = \frac{3}{2}\ln\left|\frac{1+\sin\frac{x}{3}}{1-\sin\frac{x}{3}}\right| + c$

7) $\int\limits_0^1 \arcsin\sqrt{x}\ dx = 2\int\limits_0^1 \sqrt{x}\arcsin\sqrt{x}\ dx = x^2\cdot\arcsin\sqrt{x}\Big|_0^1 - \int\limits_0^1 \frac{x^2\ dx}{\sqrt{1-x^2}} = \frac{\pi}{2} + \int\limits_0^1 \frac{1-x^2}{\sqrt{1-x^2}}\ dx - \int\limits_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} + \frac{\pi}{4} - \arcsin x\Big|_0^1 = \frac{\pi}{4}$

8) $\int \sin x\cdot\sin\ln\cos x\ dx = -\int \sin\ln\cos x\ d\cos x = -\int \sin y\ de^y \overset{def}{\equiv} -J$, where $y = \ln\cos x$.
$J = e^y\sin y - \int e^y\cos y\ dy = e^y\sin y - e^y\cos y - J$, thus $J = \frac{1}{2}e^y(\sin y - \cos y) + c$, then the initial integral equals to $-\frac{1}{2}\cos x(\sin\ln\cos x - \cos\ln\cos x) + c$

9) while $\nu(x)$ is a distance from x to the nearest integer, $\int\limits_7^{20} \nu(x)\ dx = \frac{20-7}{4} = \frac{13}{4}$

10) $\int\limits_0^{\pi/2} \mathrm{arcctg}(\tan x)\ dx = \int\limits_0^{\pi/2} \left(\frac{\pi}{2} - \mathrm{arctg}(\tan x)\right)\ dx = \frac{\pi}{2}\int\limits_0^{\pi/2} dx - \int\limits_0^{\pi/2} x\ dx = \frac{\pi^2}{4} - \frac{x^2}{2}\Big|_0^{\pi/2} = \frac{\pi^2}{8}$