

Formal asymptotics of parametric subresonance

1st Polina Astafyeva

Inst. of Math. with Com. Centre of UFRC RAS,
Ufa State Petroleum Technological University
Ufa, Russia
astafyeva.polina2@gmail.com

2nd Oleg Kiselev

Innopolis University,
Inst. of Math. with Com. Centre of UFRC RAS
Innopolis, Russia
o.kiselev@innopolis.ru

Lots of kinds of linear equations of second order are the most important models of mechanics because of Newtonian equations of a motion. Among such equations we are interested in an equation with time-dependent coefficient. In this work we consider two different equations of such kind. The first one is the equation with almost periodic coefficient:

$$u'' + (\omega^2 + \epsilon q(t))u = 0 \quad (1)$$

here $q(t)$ is almost periodic function and ϵ is a small positive parameter.

This equation has two important properties which define behaviours of solutions. The first one is a coefficient ω^2 . It defines an oscillation of the solution for the simplest case $q(t) \equiv 0$.

Another coefficient $q(t)$ can or cannot imply the resonant behaviour for the solution. The key work for understanding the behaviour of the solution of such kind of equation is the Floquet theory [1]. The parametric map for the parameters (ω, ϵ) of the equation with periodic coefficient is called Arnold's tonques [4]. Such map defines the zones of parametric resonances [5].

One of the famous examples of such equation is Mathieu equation for special kind of the coefficient:

$$q(t) \equiv a \cos(2t).$$

Although the Mathieu equation appeared in theory of Laplace equation with elliptic boundary [7] now this equation is widely used model in quantum mechanics and theory of parametric resonance [5], [6], [10], [11].

One of more general equations of such kind appeared in the theory of Moon motion [8], where $q(t)$ is a periodic function defined by their Fourier series.

Here we study more general and more natural case when the $q(t)$ is almost-periodic. Such sight on the coefficient looks as more physically important since in real phenomenon pure periodic coefficient is enveloped by additional disturbances and perturbations. Moreover, the period of the coefficient often does not coincide with period of frequency of solution for the equation, such as $q(t) \equiv 0$.

Let us consider an second order equation with almost periodic coefficient:

$$u'' + (\omega^2 + \epsilon q(t))u = 0 \quad (2)$$

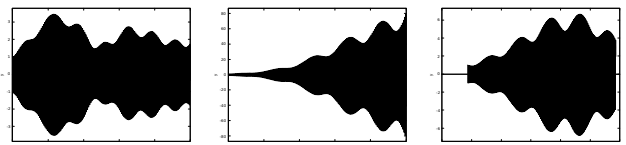


Fig. 1. Here we give three different cases for the solution of equation (2) where $k = 2$, and $p = 5$. The left picture shows bounded oscillations for the equation $\omega = 1 + \delta$, where $\delta = 0.01$, $\epsilon = 0.1$. The middle picture shows the growing solution as $\delta = 0.00347$, $\epsilon = 0.1$, and the right picture indicate the final stage of this growing solution as the same $\delta = 0.00347$, $\epsilon = 0.1$ at very large time interval around $t = 900500$.

Here almost periodic function [2], [3]:

$$q(t) = \sum_{n=1}^{\infty} \frac{1}{n^k} \cos\left(\left(2 - \frac{1}{n^p}\right)t\right). \quad (3)$$

$$k > 1, p > 0$$

The parameter ω differs little from 1: $\omega = 1 + \delta$, here δ is the parameter of the equation. The task is to determine the areas of stability of solutions of the equation depending on the parameters δ и ϵ .

In the following section we formulate aims of this work. Next section contains an asymptotic approach to the result. The last section shows the closeness of asymptotic and numerical approaches and contains the conclusions.

I. Statement of the problem

For the equation with almost-periodic coefficient one can see three different behaviour with respect to parameter $t \rightarrow \infty$.

Our first problem for studying is to understand a behaviour of solution depending on the coefficients. ω and ϵ for given form $q(t)$ as the almost-periodic function.

Second and more intrigue aim for us is to find a dependency of amplitude of the oscillated solution which one can consider as a similarity to parametric resonance growth which was derived for example in [5].

II. Asymptotic approach

Let us construct an asymptotic solution in the form:

$$u \sim u_0 + \epsilon u_1, \quad (4)$$

Let us substitute (4) in (2) and combine the terms at the same degrees ϵ . Obtain the equation for the main term:

$$\frac{d^2}{dt^2}u_0 + \omega^2 u_0 = 0.$$

Let us look for its solution in the form of $u_0 = a(\tau) \cos(\omega t) + b(\tau) \sin(\omega t)$, using two scale method [5], where $\tau = \epsilon^\gamma t$ is slow time. The parameter $\gamma > 0$, will be determined below from the condition of uniformity of the asymptotic solution for large values t .

The equation for the first correction:

$$\frac{d^2}{dt^2}u_1 + u_1 \omega^2 + q(t) (b \sin(t\omega) + a \cos(t\omega)) - 2a_1 \omega \sin(t\omega) + 2b_1 \omega \cos(t\omega) = 0 \quad (5)$$

here $a_1 = \epsilon^{\gamma-1} a'$ and $b_1 = \epsilon^{\gamma-1} b'$, a stroke means a slow-time derivative τ .

The task is to find the dependence on τ for a and b . A limited solution for the first correction of the function u_1 is constructed below.

Denote

$$f(t) = -q(t) (b \sin(t\omega) + a \cos(t\omega))$$

In this case, the general solution for the first correction equation can be written as:

$$u_1 = A \cos(\omega t) + B \sin(\omega t) - b_1 t \sin(t\omega) - a_1 t \cos(t\omega) + \frac{\cos(\omega t)}{\omega} \int_0^t f(\tilde{t}) \sin(\omega \tilde{t}) d\tilde{t} - \frac{\sin(\omega t)}{\omega} \int_0^t f(\tilde{t}) \cos(\omega \tilde{t}) d\tilde{t}. \quad (6)$$

let us introduce a replacement in the integrand function $\omega \tilde{t} \equiv \tilde{t} + \delta \tilde{t} = \tilde{t} + \kappa \tau$

Note that the formula for $f(\tilde{t})$ contains a sum over n , we change the summation with integration over \tilde{t} . Then the solution of the equation for u_1 is represented as the sum of n for the first integral in the formula (6):

$$\frac{a \sin\left(\frac{2n^p \kappa \tau + (4n^p - 1)\tilde{t}}{n^p}\right)}{4n^k} - \frac{b \cos\left(\frac{2n^p \kappa \tau + (4n^p - 1)\tilde{t}}{n^p}\right)}{4n^k} + \frac{a \sin\left(\frac{2n^p \kappa \tau + \tilde{t}}{n^p}\right)}{4n^k} - \frac{b \cos\left(\frac{2n^p \kappa \tau + \tilde{t}}{n^p}\right)}{4n^k} + \frac{b \cos\left(\frac{(2n^p - 1)\tilde{t}}{n^p}\right)}{2n^k}$$

and the second integral in the formula (6):

$$\frac{b \sin\left(\frac{2n^p \kappa \tau + (4n^p - 1)\tilde{t}}{n^p}\right)}{4n^k} + \frac{a \cos\left(\frac{2n^p \kappa \tau + (4n^p - 1)\tilde{t}}{n^p}\right)}{4n^k} + \frac{b \sin\left(\frac{2n^p \kappa \tau + \tilde{t}}{n^p}\right)}{4n^k} + \frac{a \cos\left(\frac{2n^p \kappa \tau + \tilde{t}}{n^p}\right)}{4n^k} + \frac{a \cos\left(\frac{(2n^p - 1)\tilde{t}}{n^p}\right)}{2n^k}$$

After sequential integration of all parts by \tilde{t} let us allocate the maximum order with respect to n for obtained formula :

$$\frac{\cos(\omega t)}{4\omega} \sum_{n=1}^{\infty} n^{p-k} \left(b \left(\sin(2\kappa\tau) - \sin\left(2\kappa\tau + \frac{t}{n^p}\right) \right) + a \left(\cos(2\kappa\tau) - \cos\left(2\kappa\tau + \frac{t}{n^p}\right) \right) \right)$$

and

$$\frac{\sin(\omega t)}{4\omega} \sum_{n=1}^{\infty} n^{p-k} \left(a \left(\sin(2\kappa\tau) - \sin\left(2\kappa\tau + \frac{t}{n^p}\right) \right) - b \left(\cos(2\kappa\tau) - \cos\left(2\kappa\tau + \frac{t}{n^p}\right) \right) \right)$$

In the solution u_1 the growing terms in t are combined at $\cos(\omega t)$:

$$-\frac{a}{4\omega} \left(\cos(2\kappa\tau) \sum_{n=1}^{\infty} n^{p-k} \left(\cos\left(\frac{t}{n^p}\right) - 1 \right) - \sin(2\kappa\tau) \sum_{n=1}^{\infty} \sin\left(\frac{t}{n^p}\right) \right) + \frac{b}{4\omega} \left(\sin(2\kappa\tau) \sum_{n=1}^{\infty} n^{p-k} \left(\cos\left(\frac{t}{n^p}\right) - 1 \right) + \cos(2\kappa\tau) \sum_{n=1}^{\infty} \sin\left(\frac{t}{n^p}\right) \right) - a_1 t$$

and $\sin(\omega t)$:

$$\frac{b}{4\omega} \left(\sin(2\kappa\tau) \sum_{n=1}^{\infty} n^{p-k} \sin\left(\frac{t}{n^p}\right) + \cos(2\kappa\tau) \sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{t}{n^p}\right) \right) \right) + \frac{a}{4\omega} \sin(2\kappa\tau) \left(\sum_{n=1}^{\infty} n^{p-k} \left(\cos\left(\frac{t}{n^p}\right) - 1 \right) - \cos(2\kappa\tau) \sum_{n=1}^{\infty} \sin\left(\frac{t}{n^p}\right) \right) - b_1 t$$

Replace:

$$\sum_{n=1}^{\infty} n^{p-k} \left(\cos\left(\frac{t}{n^p}\right) - 1 \right) = \sum_{n=1}^{\infty} n^{p-k} 2 \sin\left(\frac{t}{2n^p}\right)$$

In [9], the asymptotics were calculated:

$$\sum_{n=1}^{\infty} n^{p-k} \sin\left(\frac{t}{n^p}\right) \sim t^{1-\alpha} C_s, \\ C_s \sim \frac{1}{p} \int_0^{\frac{\pi}{2}} \tau^{\alpha-2} \sin(\tau) d\tau, \\ \sum_{n=1}^{\infty} n^{p-k} 2 \sin^2\left(\frac{t}{2n^p}\right) \sim t^{1-\alpha} C_c, \\ C_c \sim - \left(\frac{1}{2\pi^{1-\alpha} p (1-\alpha)} + \frac{1}{p} \int_0^{\pi} \tau^{\alpha-2} \sin^2(\tau/2) d\tau \right).$$

Denote:

$$\alpha = \frac{k-1}{p}.$$

Let us combine the growing terms in t at $\cos(\omega t)$ и $\sin(\omega t)$, in order for the coefficients with increasing summands to be zeros:

$$\begin{aligned} \epsilon^{\gamma-1} t a' &= \frac{a t^{1-\alpha}}{4\omega} (C_c \cos(2\kappa\tau) - C_s \sin(2\kappa\tau)) + \\ &+ \frac{b t^{1-\alpha}}{4\omega} (C_c \sin(2\kappa\tau) + C_s \cos(2\kappa\tau)) \end{aligned} \quad (7)$$

$$\begin{aligned} \epsilon^{\gamma-1} t b' &= \frac{b t^{1-\alpha}}{4\omega} (C_s \sin(2\kappa\tau) - C_c \cos(2\kappa\tau)) + \\ &+ \frac{a t^{1-\alpha}}{4\omega} (C_c \sin(2\kappa\tau) + C_s \cos(2\kappa\tau)) \end{aligned} \quad (8)$$

In these formulas, we will reduce both parts by t and make a replacement $t = \frac{\tau}{\epsilon^\gamma}$. As a result, we get:

$$\begin{aligned} \epsilon^{\gamma-1} \frac{d}{d\tau} a &= \frac{a \epsilon^{\gamma\alpha}}{4\tau^\alpha \omega} (C_c \cos(2\kappa\tau) - C_s \sin(2\kappa\tau)) + \\ &+ \frac{b \epsilon^{\gamma\alpha}}{4\tau^\alpha \omega} (C_c \sin(2\kappa\tau) + C_s \cos(2\kappa\tau)), \\ \epsilon^{\gamma-1} \frac{d}{d\tau} b &= \frac{b \epsilon^{\gamma\alpha}}{4\tau^\alpha \omega} (C_s \sin(2\kappa\tau) - C_c \cos(2\kappa\tau)) + \\ &+ \frac{a \epsilon^{\gamma\alpha}}{4\tau^\alpha \omega} (C_c \sin(2\kappa\tau) + C_s \cos(2\kappa\tau)). \end{aligned}$$

Hence

$$\gamma - 1 = \gamma\alpha, \quad \gamma = \frac{1}{1-\alpha}.$$

As a result, the system of equations will take the form:

$$\begin{aligned} \frac{d}{d\tau} a &= \frac{a}{4\tau^\alpha \omega} (C_c \cos(2\kappa\tau) - C_s \sin(2\kappa\tau)) + \\ &+ \frac{b}{4\tau^\alpha \omega} (C_c \sin(2\kappa\tau) + C_s \cos(2\kappa\tau)), \\ \frac{d}{d\tau} b &= \frac{b}{4\tau^\alpha \omega} (C_s \sin(2\kappa\tau) - C_c \cos(2\kappa\tau)) + \\ &+ \frac{a}{4\tau^\alpha \omega} (C_c \sin(2\kappa\tau) + C_s \cos(2\kappa\tau)). \end{aligned}$$

Denote:

$$A_\alpha = \sqrt{C_s^2 + C_c^2}, \quad \phi_\alpha = \arctan\left(\frac{C_c}{C_s}\right),$$

Then the system will take the form:

$$\begin{aligned} \frac{d}{d\tau} a &= \frac{A (b \sin(\phi + 2\kappa\tau) + a \cos(\phi + 2\kappa\tau))}{4\omega\tau^\alpha} \\ \frac{d}{d\tau} b &= \frac{A (a \sin(\phi + 2\kappa\tau) - b \cos(\phi + 2\kappa\tau))}{4\omega\tau^\alpha} \end{aligned}$$

Denote $\frac{A}{4\omega} = B$, then:

$$\frac{d}{d\tau} a = \frac{B (b \sin(\phi + 2\kappa\tau) + a \cos(\phi + 2\kappa\tau))}{\tau^\alpha}$$

$$\frac{d}{d\tau} b = \frac{B (a \sin(\phi + 2\kappa\tau) - b \cos(\phi + 2\kappa\tau))}{\tau^\alpha}$$

The system can also be written in a complex form:

$$\frac{d}{d\tau} z = \frac{B \bar{z} e^{i\phi + 2i\kappa\tau}}{\tau^\alpha}$$

$\delta = \epsilon^\gamma \kappa$

Denote $z = Z e^{i\kappa\tau + \frac{i\phi}{2}}$

$$iZ\kappa e^{\frac{i\phi}{2} + i\kappa\tau} + \left(\frac{d}{d\tau} Z\right) e^{\frac{i\phi}{2} + i\kappa\tau} = \frac{B \bar{Z} e^{\frac{i\phi}{2} + i\kappa\tau}}{\tau^\alpha}$$

simplify:

$$iZ\kappa + \frac{d}{d\tau} Z = \frac{B \bar{Z}}{\tau^\alpha}$$

$$\frac{d}{d\tau} Z = \frac{B \bar{Z}}{\tau^\alpha} - iZ\kappa$$

Let us write Z as the sum of the real and imaginary parts: $Z = w + iv$.

As a result, a system of equations is obtained:

$$\begin{aligned} \frac{d}{d\tau} w &= \frac{Bw}{\tau^\alpha} + v\kappa, \\ \frac{d}{d\tau} v &= -w\kappa - \frac{Bv}{\tau^\alpha}. \end{aligned} \quad (9)$$

III. Comparison of the asymptotic approaches and numerical results

Let us consider the properties of derived system (9) for different values of the parameters.

In the case $\kappa = 0$ the system splits into two equations. Their solutions have the following form:

$$w = C_1 \exp\left(\frac{B\tau^{1-\alpha}}{1-\alpha}\right), \quad v = C_2 \exp\left(\frac{-B\tau^{1-\alpha}}{1-\alpha}\right),$$

Here one can see the solution is exponentially growing because of $w(t)$.

For $\kappa \neq 0$ we divide both parts of the equations on the parameter κ and rewrite $\kappa\tau = \theta$. In this case we get:

$$\begin{aligned} \frac{d}{d\theta} w &= \frac{Bw}{\kappa^{1-\alpha} \theta^\alpha} + v \\ \frac{d}{d\theta} v &= -\frac{Bv}{\kappa^{1-\alpha} \theta^\alpha} - w \end{aligned}$$

Let us rewrite $B\kappa^{\alpha-1} = \lambda$, then:

$$\frac{d}{d\theta} w = \frac{\lambda}{\theta^\alpha} w + v, \quad \frac{d}{d\theta} v = -\frac{\lambda}{\theta^\alpha} v - w.$$

Here $\lambda > 0$ is a parameter of the equation and $\kappa \rightarrow 0$, $\lambda \rightarrow \infty$.

This system leads to the second order differential equation:

$$\frac{d^2}{d\theta^2} w + \left(1 - \frac{\lambda^2}{\theta^{2\alpha}} + \frac{\alpha\lambda}{\theta^{\alpha+1}}\right) w = 0$$

Here λ is a large parameter. So asymptotic solution can be obtained by the WKB method [12]:

$$w \sim C_1 \frac{\exp(\int^t \sqrt{1 - \frac{\lambda^2}{\theta^{2\alpha}} + \frac{\alpha\lambda}{\theta^{\alpha+1}}} dt)}{\sqrt[4]{1 - \frac{\lambda^2}{\theta^{2\alpha}} + \frac{\alpha\lambda}{\theta^{\alpha+1}}}}.$$

So the turning point to change the growing character of the solution is located in a neighbourhood of the point

$$\theta_* \sim \lambda^{1/\alpha}.$$

Let us compare the position of the tuning point obtained here by asymptotical way and the results which were shown in the figure.

For the numerical example we have following data

$$k = 2, \quad p = 5, \quad \alpha \equiv (k-1)/p = 1/5, \quad \gamma \equiv 1/(1+\alpha) = 6/5, \\ A \sim 1.09264275,$$

here the value of A was obtained with respect to the work [9].

$$B \equiv \frac{A}{4\omega} \sim 1/4, \quad \lambda = 5, \quad \kappa \equiv (B/\lambda)^{1/(1-\alpha)} \sim \left(\frac{1}{4}\right)^{5/4}, \\ \epsilon = 0.1, \quad \delta = \epsilon^{\gamma\kappa}, \quad \theta_* \sim \lambda^{1/\alpha}, \quad T = \frac{\theta_*}{\kappa\epsilon^\gamma} \sim 900473.$$

Conclusions. We obtain the system of equations for parametric sub-resonant growth of the amplitude of oscillations. The growth depends on the slow variable $\tau = \epsilon^\gamma t$. We find also the time of turning point from the growing of the amplitude to the bounded oscillations in the slow variable τ . The comparison between the asymptotic approximation for the turning time and numerical one is shown.

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