On the stability of the system of Thomson vortex n-gon and a moving circular cylinder

Leonid Kurakin
Water Problems Institute, RAS
Moscow, Russia
Southern Mathematical Institute, VSC RAS
Vladikavkaz, Russia
Southern Federal University
Rostov on Don, Russia
lgkurakin@sfedu.ru

Irina Ostrovskaya
Southern Federal University
Rostov on Don, Russia
ivostrovskaya@sfedu.ru

***Abstract*—The stability problem of a moving circular cylinder of radius R and a system of n identical point vortices uniformly distributed on a circle of radius R_0 is considered. The circulation around the cylinder is zero. There are three parameters in the problem: the number of point vortices n, the added mass of the cylinder a and parameter $q = R^2/R_0^2$.**

The linearization matrix and the quadratic part of the Hamiltonian of the problem are studied. As a result, the parameter space of the problem is divided into the area of linear stability, where nonlinear analysis is required, and the instability area. In the case $n = 2, 3$ two domains of linear stability are found. In the case $n = 4, 5, 6$ there is found one domain. In the case $n \geqslant 7$, the studied solution is unstable for all of problem parameters values. The obtained results in the limiting case at $a \to \infty$ agree with the known results for the model of point vortices outside the circular domain.

***Index Terms*—Point vortices, Hamiltonian equation, Thomson polygon, Stability**

I. Introduction

Thomson vortex polygon is a configuration of identical point vortices located at the vertices of a regular polygon. This vortex configuration owes its name to two famous scientists. W. Thomson (Lord Kelvin) posed the stability problem of such a polygon on the plane in connection with his vortex theory of the atom [1]. Its study in a linear formulation was began by J. J. Thomson [2] and completed by T. H. Havelock [3]. The history of solving this problem in linear and nonlinear formulation is described in detail in [4], [5].

The numerous studies have been devoted to the dynamics of point vortices outside a circular domain (see review [6]). The stability problem of stationary rotation of Thomson vortex polygon outside unmoving circular cylinder with the zero circulation around the cylinder had been solved by Havelock in [3]. The nonlinear analysis of this problem required the involvement of the resonances theory of equilibria of Hamiltonian systems (see review [7]). It turned out that two resonances lead to instability, although stability takes place in the linear formulation. The effect of circulation in the problem under consideration in the case of a vortices outside a circle was studied in [8].

Various forms of the motion equations for a moving rigid circular cylinder interacting with an n point vortices were obtained in [9]–[13]. The history of the derivation of these equations is given in the introduction of [13].

In this paper, the stability of a system consisting of a Thomson n-gon and a moving cylinder is studied for arbitrary n with zero circulation around cylinder. A linear stability analysis is carried out for an arbitrary number of point vortices $n \geqslant 2$. In the case $n \leqslant 6$, linear stability conditions are found under which nonlinear analysis is required to solve the stability problem. It is prooved that in the case of $n \geqslant 7$, the considered system is unstable for all values of the problem parameters. Here we also correct the erroneous results of the paper [14] for the case $n = 2$.

II. Formulation of problem

The motion of a circular cylinder interacting with n identical point vortices is considered. As a result of the reduction of the complete equations of motion, system of equations are written in the complex form in [12]

$$\begin{aligned} a\dot{z}_c &= av = -i\gamma z_c + i\gamma_0 \sum_{j=1}^{n} (\widetilde{z}_j - z_j), \\ \dot{\overline{z}}_k &= -\overline{v} + \frac{R^2 v}{z_k^2} + \frac{i\gamma}{z_k} - \frac{i\gamma_0}{z_k - \widetilde{z}_k} + \\ &+ i\gamma_0 \sum_{\substack{j=1\\ j\neq k}}^{n} \left(\frac{1}{z_k - z_j} - \frac{1}{z_k - \widetilde{z}_j} \right), \quad k = 1, \dots, n. \end{aligned} \tag{1}$$

Here complex variables $z_c = x_c + iy_c$, $z_k = x_k + iy_k$ define the position of the cylinder and point vortices, $v = v_1 + iv_2$ is cylinder velocity, $\widetilde{z}_k = \dfrac{R^2}{\overline{z}_k}$ is the reflection of the kth vortex from the boundary of the circle, R is cylinder radius, the constant coefficient a involves the added mass of the cylinder, and the constants γ and γ_0 are connected with the circulation

The work of the first author was carried out within the framework of the project no. FMWZ-2022-0001 of the State Task of the IWP RAS

around the cylinder Γ and the intensity of identical point vortices Γ_0 by the formulae $\gamma = \frac{\Gamma}{2\pi}$ and $\gamma_0 = \frac{\Gamma_0}{2\pi}$.

Further in the paper we will consider the case

$$\gamma = n\gamma_0. \tag{2}$$

Note that in [15] a complete bifurcation analysis of the motion of the circular cylinder and two point vortices with arbitrary circulation was carried out in the case not considered here when circulation γ and the total impulse of the system are equal to zero. In [13] this works has been made for case of two point vortices with opposite intensity.

The system (1), (2) can be written in the form

$$n\gamma_0 \dot{z}_c = -2iH_{\overline{z}_c}, \qquad \gamma_0 \dot{\overline{z}}_k = -2iH_{z_k}. \tag{3}$$

The Hamiltonian $H = H(\mathbf{z}, \overline{\mathbf{z}})$, $\mathbf{z} = (z_c, z_1, \dots, z_n)$ is given by formula

$$\begin{aligned} H = \frac{1}{2}a|v|^2 + \frac{\gamma_0^2}{2}\sum_{k=1}^{n}\left(\ln\left(|z_k|^2 - R^2\right) - n\ln|z_k|^2\right) + \\ + \frac{\gamma_0^2}{2}\sum_{1\le k<j\le n}\left(\ln\left|R^2 - z_j\overline{z}_k\right|^2 - \ln|z_j - z_k|^2\right). \end{aligned} \tag{4}$$

The system (3) has the solution

$$z_c = 0, \quad z_k = e^{i\omega_n t}u_k, \quad u_k = R_0 e^{i\frac{2\pi}{n}(k-1)} \tag{5}$$

corresponding of the stationary rotation of the n point vortices around the cylinder with constant angular velocity ω_n:

$$\omega_n = -\frac{\gamma_0}{2R_0^2}\left(3n - 1 - \frac{2n}{1-q^n}\right), \qquad q = \frac{R^2}{R_0^2}. \tag{6}$$

The point vortices are located uniformly on a circle of radius R_0, $R_0 > R$. Then $0 < q < 1$.

Without loss of generality, we will further assume that

$$\gamma_0 = 1, \quad R_0 = 1. \tag{7}$$

The change of variables

$$\begin{aligned} &z_c = (q_c + ip_c)e^{i\omega_n t}, \\ &z_k = \sqrt{1+2r_k}\,e^{i(\omega_n t + \frac{2\pi(k-1)}{n} + \theta_k)}, \quad k = 1, \dots, n, \end{aligned} \tag{8}$$

reduces the system (3), (4) to the perturbation equations with the Hamiltonian $E = E(\boldsymbol{\rho})$, $\boldsymbol{\rho} = (q_c, r_1, \dots, r_n, p_c, \theta_1, \dots, \theta_n)$:

$$\begin{aligned} E(\boldsymbol{\rho}) = H(\mathbf{z}(\boldsymbol{\rho}), \overline{\mathbf{z}}(\boldsymbol{\rho})) + \\ + \frac{\omega_n}{2}\left(nq_c^2 + np_c^2 - n - 2\sum_{k=1}^{n} r_k\right). \end{aligned} \tag{9}$$

The stationary solution (5) corresponds to a continuous family of equilibria

$$\mathcal{C} = \{q_c = p_c = r_1 = \ldots = r_n = 0,\ \theta_1 = \ldots = \theta_n\}. \tag{10}$$

The stability of the continuous family of equilibria (10) is equivalent to the orbital stability of the stationary solution (5).

Let

$$E_2(\boldsymbol{\rho}) = \langle \mathbf{S}_n \boldsymbol{\rho}, \boldsymbol{\rho}\rangle \tag{11}$$

be the quadratic terms of the expansion of the Hamiltonian $E(\boldsymbol{\rho})$ into Taylor series in power of $\boldsymbol{\rho}$ in a neighborhood of the zero equilibrium position. Here $\langle\cdot,\cdot\rangle$ is the scalar product.

The symmetric matrix $\mathbf{S}_n$ has the form

$$\mathbf{S}_n = \begin{pmatrix} na_{1n} & nb_{1n}\mathbf{h}_1 & 0 & -nb_{2n}\mathbf{h}_{n-1} \\ nb_{1n}\mathbf{h}_1 & \mathbf{F}_{1n} & nb_{1n}\mathbf{h}_{n-1} & \mathbf{G}_{0n} \\ 0 & nb_{1n}\mathbf{h}_{n-1} & na_{1n} & nb_{2n}\mathbf{h}_1 \\ -nb_{2n}\mathbf{h}_{n-1} & -\mathbf{G}_{0n} & nb_{2n}\mathbf{h}_1 & \mathbf{F}_{2n} \end{pmatrix}$$

The vectors $\mathbf{h}_1$ and $\mathbf{h}_{n-1}$ are given by formulas

$$\begin{aligned} \mathbf{h}_1 &= \sqrt{\frac{2}{n}}(1, \cos\nu, \dots, \cos(n-1)\nu), \\ \mathbf{h}_{n-1} &= \sqrt{\frac{2}{n}}(0, \sin\nu, \dots, \sin(n-1)\nu), \end{aligned} \tag{12}$$

Here $\nu = \frac{2\pi}{n}$, coefficients a_{1n}, b_{1n} and b_{2n} are given by formulas

$$a_{1n} = \frac{n}{2a} + \frac{\omega_n}{2},\ b_{1n} = \frac{\sqrt{2n}(1+q)}{4a},\ b_{2n} = \frac{\sqrt{2n}(1-q)}{4a}.$$

Symmetric matrices $\mathbf{F}_{1n}$, $\mathbf{F}_{2n}$ and skew-symmetric matrix $\mathbf{G}_{0n}$ are circulant matrices:

$$\mathbf{F}_{mn} \overset{def}{=} f_{m0}\mathbf{I}_n + \sum_{j=1}^{n-1} f_{mj}\mathfrak{C}^j,\ \mathbf{G}_0 \overset{def}{=} \sum_{j=1}^{n-1} g_{0j}\mathfrak{C}^j,$$

$$\mathfrak{C} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where $\mathbf{I}_n$ is unit matrix of size $n \times n$.

The coefficient f_{mj} and g_{0j} can be written as:

$$\begin{aligned} f_{m0}(q,a) &= \frac{1}{2}f_{m0}^0(q) + f_{m0}^1(a,q), \\ f_{mj}(q,a) &= \frac{1}{2}f_{mj}^0(q) + f_{mj}^1(a,q), \\ g_{0j}(q,a) &= \frac{1}{2}g_{0j}^0(q) + g_{0j}^1(a,q), \quad j = 1, \dots, n-1. \end{aligned} \tag{13}$$

Here the coefficients f_{mj}^0 and g_{0j}^0 match the corresponding coefficients written out in the papers [3], [8].

The coefficients f_{mj}^1 and g_{0j}^1, $j = 0, \dots, n-1$ are given by equalities

$$\begin{aligned} f_{1j}^1(a,q) &= \frac{(1+q)^2}{2a}\cos\nu j,\ j = 0, \dots, n-1 \\ f_{2j}^1(a,q) &= \frac{(1-q)^2}{2a}\cos\nu j, \\ g_{0j}^1(a,q) &= \frac{q^2-1}{2a}\sin\nu j. \end{aligned} \tag{14}$$

The values λ_{mk} and $i\lambda_{0k}$ are eigenvalues of matrices $\mathbf{F}_{mn}$ and $\mathbf{G}_{0n}$ given in the form

$$\lambda_{mk}(q,a) = \frac{1}{2}\lambda_{mk}^0(q) + \lambda_{mk}^1(q,a), \tag{15}$$

where λ^0_{1k}, λ^0_{2k} and λ^0_{0k} are the same as in the papers [3], [8] and λ^1_{1k}, λ^1_{2k} and λ^1_{0k} are defined as

$$\begin{aligned}
\lambda^1_{11} &= \lambda^1_{1,n-1} = \frac{n(1+q)^2}{4a},\\
\lambda^1_{21} &= \lambda^1_{2,n-1} = \frac{n(1-q)^2}{4a},\\
\lambda^1_{01} &= -\lambda^1_{0,n-1} = \frac{n(q^2-1)}{4a},\\
\lambda^1_{1k} &= \lambda^1_{2k} = \lambda^1_{0k} = 0, \quad k \neq 1, n-1.
\end{aligned} \tag{16}$$

The symmetric matrix $\mathbf{S}_n$ has zero eigenvalue, corresponding to family of equilibrium (10). The family $\mathcal{C}$ is Lyapunov stable in the exact nonlinear setting if the Hamiltonian $E(\boldsymbol{\rho})$ has an extremum on it. To do this, it is necessary that all the eigenvalues of the matrix $\mathbf{S}_n$ except for a simple zero have the same sign. As shown later in the Proposition 2, this does not hold.

The linearization matrix $\mathbf{L}_n$ of the system with Hamiltonian (9) about the zero solution has the form

$$\mathbf{L}_n = 2\mathbf{K}^{-1}\mathbf{J}\mathbf{S}_n, \qquad \mathbf{J} = \begin{pmatrix} 0 & \mathbf{I}_{n+1} \\ -\mathbf{I}_{n+1} & 0 \end{pmatrix}, \tag{17}$$

where $\mathbf{K}^{-1}$ is diagonal matrix of size $(2n+2)\times(2n+2)$:

$$\mathbf{K}^{-1} = \operatorname{diag}\left(1/n, -1, \ldots, -1, 1/n, -1, \ldots, -1, \right).$$

Then matrix $\mathbf{L}_n$ can write as

$$\mathbf{L}_n = 2\begin{pmatrix} 0 & b_{1n}\mathbf{h}_{n-1} & a_{1n} & b_{1n}\mathbf{h}_1 \\ nb_{2n}\mathbf{h}_{n-1} & \mathbf{G}_{0n} & -nb_{1n}\mathbf{h}_1 & -\mathbf{F}_{2n} \\ -a_{1n} & -b_{1n}\mathbf{h}_1 & 0 & b_{2n}\mathbf{h}_{n-1} \\ nb_{1n}\mathbf{h}_1 & \mathbf{F}_{1n} & nb_{1n}\mathbf{h}_{n-1} & \mathbf{G}_{0n} \end{pmatrix}.$$

Instability of solution (5) occurs when the linearization matrix $\mathbf{L}_n$ has eigenvalues with positive real part.

In the case $n = 2$ the characteristic polynomial of the linearization matrix $\mathbf{L}_2$ is given by equality

$$\det(\sigma\mathbf{I}_6 - \mathbf{L}_2) = \sigma^2\Psi_{12}(\sigma^2), \tag{18}$$

where $\Psi_{12}(\sigma^2)$ is biquadratic polynomial

$$\Psi_{12}(\sigma^2) = \sigma^4 + 8p_{11}\sigma^2 + 16p_{01}, \tag{19}$$

$$p_{11} = a_{12}^2 - 8b_1b_2 + \lambda_{11}\lambda_{21}, \tag{20}$$

$$p_{01} = a_{12}^2\lambda_{11}\lambda_{21} - 4b_{12}^2\lambda_{21} - 4b_{22}^2\lambda_{11} + 16b_{12}^2b_{22}^2. \tag{21}$$

If $n \geqslant 3$, the eigenvalues of the linearization matrix $\mathbf{L}_n$ are root of the following polynomial:

$$\det(\sigma\mathbf{I}_{2n+2} - \mathbf{L}_n) = \sigma^2\prod_{j=1}^{\lfloor\frac{n}{2}\rfloor}\Psi_{jn}(\sigma^2). \tag{22}$$

Here $\Psi_{1n}(\sigma^2)$ is bicubic polynomial

$$\begin{aligned}
&\Psi_{1n}(\sigma^2) = \sigma^6 + 4p_{21}\sigma^4 + 16p_{11}\sigma^2 + 64p_{01} \qquad (23)\\
&p_{21} = 2\,\lambda_{11}\lambda_{21} + 2\,\lambda_{01}^2 - 4\,nb_{1n}b_{2n} + a_{1n}^2\\
&p_{11} = 4n^2b_{1n}^2b_{2n}^2 - 4n\lambda_{01}^2b_{1n}b_{2n} - 4n\lambda_{01}\lambda_{21}b_{1n}^2\\
&\quad - 4n\lambda_{01}\lambda_{11}b_{2n}^2 + 4n\lambda_{01}a_{1n}b_{1n}b_{2n} - 4n\lambda_{21}\lambda_{11}b_{1n}b_{2n}\\
&\quad - 2n\lambda_{21}a_{1n}b_{1n}^2 - 2n\lambda_{11}a_{1n}b_{2n}^2 + \lambda_{01}^4 - 2\lambda_{01}^2\lambda_{21}\lambda_{11}\\
&\quad + 2\lambda_{01}^2a_{1n}^2 + \lambda_{21}^2\lambda_{11}^2 + 2\lambda_{21}\lambda_{11}a_{1n}^2\\
&p_{01} = \big(2n\,b_{1n}b_{2n}\,\lambda_{01} + nb_{1n}^2\lambda_{21} + n\lambda_{11}b_{2n}^2 + a_{1n}\lambda_{01}^2\\
&\quad -\lambda_{11}\lambda_{21}a_{1n}\big)^2,
\end{aligned}$$

the polinomial $\Psi_{jn}(\sigma^2)$, $j = 2, \ldots, \lfloor\frac{n-1}{2}\rfloor$ for $n > 3$ are biquadratic polynomials

$$\begin{aligned}
&\Psi_{jn}(\sigma^2) = \sigma^4 + 8p_{1j}\sigma^2 + 16p_{0j} \qquad (24)\\
&p_{1j} = \lambda_{1j}\,\lambda_{2j} + \lambda_{0j}^2, \quad p_{0j} = \left(\lambda_{1j}\lambda_{2j} - \lambda_{0j}^2\right)^2.
\end{aligned}$$

In the case of even $n \geqslant 4$ the polynomial $\Psi_{\frac{n}{2},n}(\sigma^2)$ has the form

$$\Psi_{\frac{n}{2},n}(\sigma^2) = \sigma^2 + 4\lambda_{11}\lambda_{12}. \tag{25}$$

In the case $n \geqslant 4$, the polynomial (24), (25) do not depend on parameter a, and their roots coincide with the corresponding eigenvalues of the linearization matrix in the case of a unmoving cylinder [3]. In [3] it has been shown that among them for $n \geqslant 7$ there is at least one eigenvalue with positive real part. Hence, and from the analysis of polynomials Ψ_{jn} in the case of $2 \leqslant n \leqslant 6$, the following statements follow.

Proposition 1.

1) *If $n \geqslant 7$ the solution* (5), (6) *is unstable for any values of problem parameters: $0 < q < 1$, $a > 0$.*
2) *If $n = 2$ all eigenvalues of the linearization matrix $\mathbf{L}_2$ lie in the imaginary axes, if coefficients of biquadratic polynomial $\Psi_{12}(\sigma^2)$ satisfy to conditions:*

$$\begin{aligned}
p_{01}(q,a) \geqslant 0, \quad p_{11}(q,a) \geqslant 0,\\
\mathcal{D}_{12}(q,a) = p_{11}^2 - 4p_{01} \geqslant 0.
\end{aligned} \tag{26}$$

3) *If $3 \leqslant n \leqslant 6$, all eigenvalues of the linearization matrix $\mathbf{L}_n$ lie in the imaginary axes, if all following conditions are valid:*
   a) *The cubic polynomial $\Psi_{1n}(s)$ is stable, that is, satisfies the Vyshnegradsky conditions:*

$$\begin{aligned}
&p_{01}(q,a) \geqslant 0,\ p_{11}(q,a) \geqslant 0,\ p_{21}(q,a) \geqslant 0,\\
&\Delta_{1n}(q,a) = p_{11}p_{21} - p_{01} \geqslant 0. \qquad (27)
\end{aligned}$$

   and its discriminant $\mathcal{D}_{1n}$ is not negative:

$$\begin{aligned}
\mathcal{D}_{1n}(q,a) = -4p_{21}^3p_{01} + p_{21}^2p_{11}^2 - 4p_{11}^3 +\\
+ 18p_{01}p_{11}p_{21} - 27p_{01}^2 \geq 0.
\end{aligned} \tag{28}$$

   b) *The coefficients and discriminant of quadratic polynomials $\Psi_{jn}(s)$, $j = 2, \ldots, \lfloor\frac{n}{2}\rfloor$ for $n > 3$ are not negative:*

$$\begin{aligned}
p_{0j}(q) \geqslant 0, \quad p_{1j}(q) \geqslant 0,\\
\mathcal{D}_{jn}(q) = p_{1j}^2 - 4p_{0j} \geqslant 0.
\end{aligned} \tag{29}$$

4) *The linearization matrix $\mathbf{L}_n$, $n = 2, \ldots, 6$ has at least one eigenvalue in the right half-plane, if at least one of the conditions 2) or 3) is violated.*

Let's analyze the eigenvalues of the matrix $\mathbf{S}_n$ in the case $2 \leqslant n \leqslant 6$.

If $n = 2$, then its characteristic polynomial has the form

$$\det(\Lambda \mathbf{I}_6 - \mathbf{S}_2) = \Lambda(\Lambda - \lambda_{12})\Phi_{21}(\Lambda)\Phi_{22}(\Lambda),$$

where $\Phi_{2m}(\Lambda)$ are quadratic polynomials

$$\Phi_{2m} = \Lambda^2 + k_{1m}\Lambda + k_{0m} = 0,$$
$$k_{1m} = -2a_{12} + \lambda_{m1}, \quad k_{0m} = 8b_{m2}^2 - 2a_{12}\lambda_{m1}.$$

In the case $n = 3, 5$ the characteristic polynomials of the matrix $\mathbf{S}_n$ are

$$\begin{aligned} \det(\Lambda \mathbf{I}_8 - \mathbf{S}_3) &= \Lambda(\Lambda - \lambda_{13})\Phi_{31}^2(\Lambda), \\ \det(\Lambda \mathbf{I}_{12} - \mathbf{S}_5) &= \Lambda(\Lambda - \lambda_{15})\Phi_{51}^2(\Lambda)\Phi_{52}^2(\Lambda). \end{aligned} \tag{30}$$

In the case $n = 4$, 6, we have

$$\begin{aligned} \det(\Lambda \mathbf{I}_{10} - \mathbf{S}_4) &= \Lambda(\Lambda - \lambda_{14})\Phi_{41}^2(\Lambda)\Phi_{42}(\Lambda), \\ \det(\Lambda \mathbf{I}_{14} - \mathbf{S}_6) &= \Lambda(\Lambda - \lambda_{16})\Phi_{61}^2(\Lambda)\Phi_{62}^2(\Lambda)\Phi_{63}(\Lambda). \end{aligned} \tag{31}$$

The polynomials $\Phi_{n1}(\Lambda)$ in (30), (31) are cubic polynomials given by formula

$$\begin{aligned} &\Phi_{n1}(\Lambda) = \Lambda^3 + k_{21}\Lambda^2 + k_{11}\Lambda + k_{01}, \\ &k_{21}(q, a) = -n\, a_{1n} - \lambda_{21} - \lambda_{11}, \\ &k_{11}(q, a) = -\left(b_{1n}^2 + b_{2n}^2\right) n^2 + a_{1n}\left(\lambda_{21} + \lambda_{11}\right) n \\ &\qquad + \lambda_{21}\lambda_{11} - \lambda_{01}^2, \\ &k_{01}(q, a) = \left(2\,\lambda_{01} b_{1n}\, b_{2n} + \lambda_{11} b_{2n}^2 + \lambda_{21} b_{1n}^2\right) n^2 \\ &\qquad - a_{1n}\left(\lambda_{21}\lambda_{11} - \lambda_{01}^2\right) n. \end{aligned} \tag{32}$$

The polynomial $\Phi_{42}(\Lambda)$ and $\Phi_{63}(\Lambda)$ are written as

$$\Phi_{n,\frac{n}{2}}(\Lambda) = (\Lambda - \lambda_{1,\frac{n}{2}})(\Lambda - \lambda_{2,\frac{n}{2}}).$$

The polynomials $\Phi_{52}(\Lambda)$ and $\Phi_{62}(\Lambda)$ are quadratic polinomial:

$$\begin{aligned} &\Phi_{n2}(\Lambda) = \Lambda^2 + k_{12}\Lambda + k_{02}, \\ &k_{12}(q) = -\lambda_{12} - \lambda_{22}, \qquad k_{02}(q) = \lambda_{12}\lambda_{22} - \lambda_{02}^2. \end{aligned} \tag{33}$$

The calculations show that in the case $3 \leqslant n \leqslant 6$ the roots of cubic polynomials $\Phi_{n1}(\Lambda)$ have different signs. Similar case takes place for polynomial $\Phi_{21}\Phi_{22}$ at $n = 2$.

Proposition 2. *The matrix $\mathbf{S}_n$, $2 \leqslant n \leqslant 6$, has eigenvalues with different sings for all values of the problem parameter and quadratic form* (11) *is sign-variable one. Thus nonlinear stability analysis is required in linear stability domains (white domain in the Figs 1–3).*

III. Formulation of results

In the case $n \geqslant 7$ the solution (5), (6) is unstable for all of values of problem parameter (Proposition 1).

We introduce the parameter α:

$$\alpha = \frac{1 - a}{1 + a}. \tag{34}$$

In the case $2 \leqslant n \leqslant 6$, analysis of the eigenvalues of the matrices $\mathbf{L}_n$ and $\mathbf{S}_n$ show that the space of parameters (q, α), where $0 < q < 1$, and $-1 < \alpha < 1$, is divided into two types of areas shown in the Figs. 1–3:

1) White area is linear stability area, where the eigenvalues of linearization matrix $\mathbf{L}_n$ are all purely imaginary, and the eigenvalues of the matrix $\mathbf{S}_n$ have the different signs. In this case, a nonlinear analysis is required to conclude stability.
2) Shaded area is instability area. The linearization matrix $\mathbf{L}_n$ has eigenvalues with positive real part. The solution (5), (6) is instable in the exact nonlinear setting.

The Figs. 1, 2 present the cases $n = 2, 3$ respectively, and the Fig. 3 shows the cases $n = 4, 5, 6$.

For $n = 2, 3$ two areas of linear stability are found. In the Fig. 1 the curves α_{12} are given by equation $\mathcal{D}_{12} = 0$, where $\mathcal{D}_{12}$ is discriminant of biquadratic polynomial (19). The curves α_{22} are defined by equality $p_{01} = 0$, where p_{01} is given by (21). In the Fig. 2 the boundary of linear stability domains are given by equality $\mathcal{D}_{13} = 0$, where $\mathcal{D}_{13}$ is given by formula (28).

In the cases $n = 4, 5, 6$ one linear stability area is shown in Fig. 3. Its boundary consists of the curve α_{1n}, given by the equation $\mathcal{D}_{1n} = 0$ and straight line $q = q_{*n}$. Here $\mathcal{D}_{1n}$ is given by equality (28). The constants q_{*n} was calculate by Havelock in [3]:

$$q_{*2} \approx 0.148536, \quad q_{*3} \approx 0.273695, \quad q_{*4} \approx 0.308125,$$
$$q_{*5} \approx 0.334596, \quad q_{*6} \approx 0.295985.$$

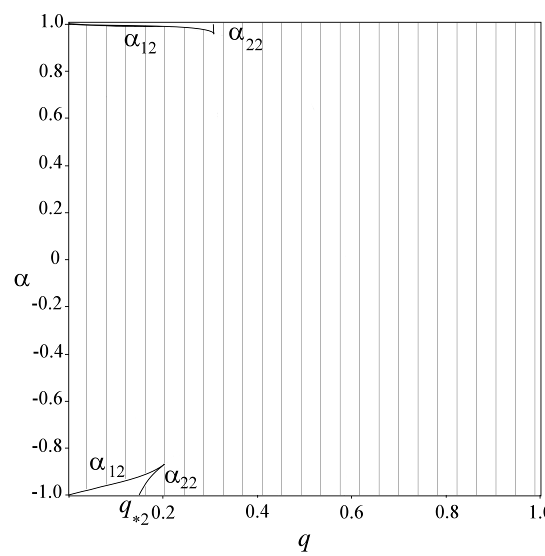

Fig. 1. The diagram of stability of stationary rotation of the Thomson vortex n-gon around cylinder (the solution (5)) in the case $n = 2$

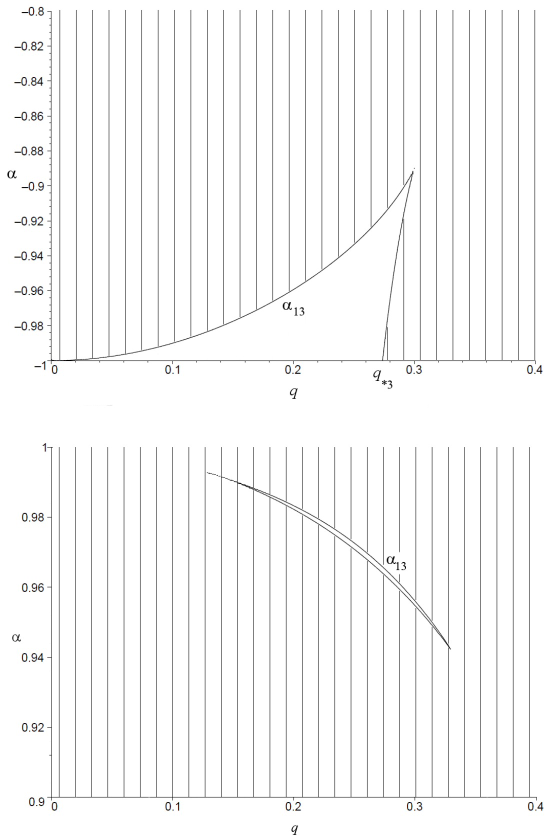

Fig. 2. The diagram of stability of the solution (5) in the case $n = 3$

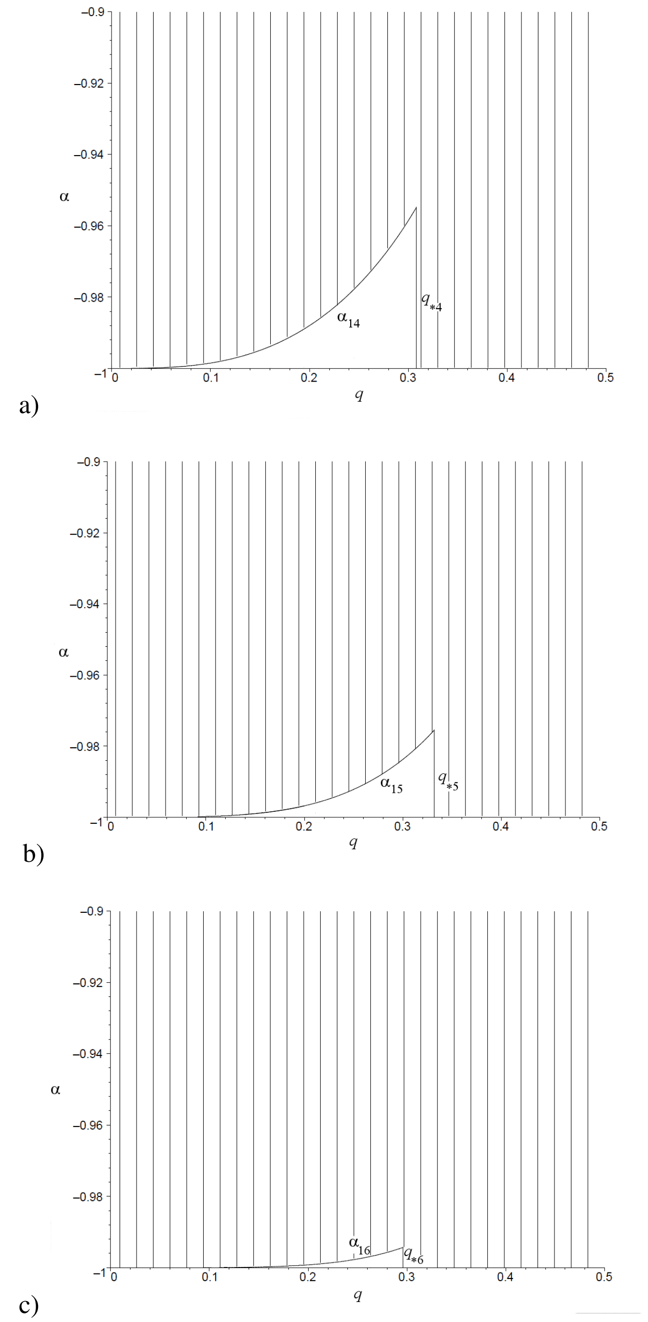

Fig. 3. The diagram of stability of the solution (5) in the case a) $n = 4$, b) $n = 5$, c) $n = 6$

REFERENCES

[1] W. Thomson, “Floating magnets (illustrating vortex systems)”, Nature, London, vol. 18, pp. 13–14, 1878.

[2] J. J. Thomson, A Treatise on the Motion of Vortex Rings: An essay to which the Adams Prize was adjudged in 1882, in the University of Cambridge. London: Macmillan, pp. 94–108, 1883.

[3] T. H. Havelock, “The stability of motion of rectilinear vortices in ring formation”, Phil. Mag., vol. 11, no 70, pp. 617-633, 1931.

[4] L. G. Kurakin and V. I. Yudovich, The stability of stationary rotation of a regular vortex polygon, Chaos, vol. 12, pp. 574–595, 2002.

[5] A. V. Borisov and I. S. Mamaev, “The Kelvin problem and its solutions”, in Mathematical Methods in the Dynamics of Vortex Structures. Moscow–Izhevsk, Institute of Computer Sciences, pp. 261–265, 2005.

[6] A. A. Kilin, A. V. Borisov, and I. S. Mamaev, “The Dynamics of Point Vortices Inside and Outside a Circular Domain”, Fundamental and Applied Problems in the Theory of Vortices, pp. 414–440, 2003.

[7] L. G. Kurakin, A. P. Melekhov, and I. V. Ostrovskaya, “A Survey of the Stability Criteria of Thomson’s Vortex Polygons outside a Circular Domain”, Bol. Soc. Mat. Mexicana, vol. 22. N 2. P. 733-744, 2016.

[8] L. Kurakin and I. Ostrovskaya “On the Effects of Circulation around a Circle on the Stability of a Thomson Vortex N-gon”, Mathematics, vol. 8. p. 1033, 2020.

[9] S. M. Ramodanov, “Motion of a circular cylinder and N point vortices in a perfect fluid,” Regular Chaotic Dyn., vol. 7, pp. 291, 2002.

[10] B. N. Shashikanth, J. E. Marsden, J. W. Burdick, and S. D. Kelly, “The Hamiltonian structure of a two-dimensional rigid circular cylinder interacting dynamically with N point vortices,” Phys. Fluids, vol. 14, pp. 1214, 2002.

[11] A. V. Borisov, I. S. Mamaev, and S. M. Ramodanov, “Motion of a circular cylinder and N point vortices in a perfect fluid,” Regular Chaotic Dyn. vol. 8, pp. 449, 2003.

[12] A. V. Borisov, I. S. Mamaev, S. M. Ramodanov, “Dynamics of a cylinder interacting with point vortices”, in Mathematical Methods in the Dynamics of Vortex Structures. Moscow–Izhevsk, Institute of Computer Sciences, pp. 286–307, 2005.

[13] I. S. Mamaev, I. A. Bizyaev, “Dynamics of an unbalanced circular foil and point vortices in an ideal fluid”, Phys. Fluids, vol. 33, 087119, 2021.

[14] A. V. Borisov and L. G. Kurakin, “On the Stability of a System of Two Identical Point Vortices and a Cylinder”, Proc. Steklov Inst. Math., vol. 310, pp. 25–31, 2020.

[15] S. M. Ramodanov and S. V. Sokolov, “Dynamics of a Circular Cylinder and Two Point Vortices in a Perfect Fluid”, Regular Chaotic Dyn., vol. 26, no. 6, pp. 675–691, 2021.